

Lecture 3 (Feb 8, 2016)

Last time we looked at phase portraits of linear systems $\dot{x} = Ax$.

- How conclusive linearization approach depends on how phase portrait of linear systems persist under perturbations.

Consider "linear" perturbations $A \mapsto A + \Delta A$

useful fact: Eigenvalues of A depend continuously on elements of A .

Given $\varepsilon > 0$, $\exists \delta > 0$ st if $|\bar{a}_{ij} - a_{ij}| < \delta$, $\forall i, j$, then $|\bar{\lambda}_i - \lambda_i| < \varepsilon$,

for all $\lambda_i \in \text{spec}(A)$. (bar denotes the perturbed one)

\Rightarrow under small perturbations, LHP eig's remain LHP eig's

$$\begin{pmatrix} \text{LHP: left-half plane} \\ \text{RHP: right-half plane} \end{pmatrix} \quad \text{RHP} = = = \text{RHP} =$$

Stable, unstable node & focus & saddle points are said to be "structurally stable", because they maintain their qualitative behavior under small perturbations.

Repeated nonzero real eigenvalues can become a pair of complex eigenvalues under small perturbations. Hence a stable (unstable) node would either remain a stable (unstable) node or become a stable (unstable) focus.

However, imaginary axis eigenvalues can be perturbed off the imaginary axis:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \lambda_{1,2} = \pm i \quad \begin{pmatrix} 8 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \lambda_{\pm} = \frac{8 \pm \sqrt{8^2 - 4}}{2}$$

\Rightarrow a center equilibrium point is not structurally stable.

- When there is one eigenvalue at $\lambda_1=0$, Perturbation results in 2 real distinct eigenvalues. $|\lambda_1-\mu| \ll |\lambda_2|$ can get node or Saddle point:
 \uparrow
 size of perturbation

For $\lambda_2 < 0$, trajectories will converge relatively fast to v_1 . slow behavior along v_1 is governed by sign of μ . (Singularly perturbed systems, chap 11)

- When there are two zero eigenvalues & $A \neq 0$, Perturbation can get any type of phase portrait.

Def. An equilibrium point $x=x^*$ of $\dot{x}=f(x)$, $x \in \mathbb{R}^n$, is a "hyperbolic eq pt" if the linearization A has no eigenvalues with zero real part.

Example of inconclusiveness of imaginary axis eigenvalues:

$$(i) \quad \dot{x} = -x^3$$

$$x^* = 0, \quad x(0) \rightarrow 0 \quad \text{if } x(0) \neq 0$$

but linearization at $x^*=0$ is $\dot{z}=0$ which does not give this conclusion.

$$(ii) \quad \begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - \varepsilon x_1^2 x_2 \end{aligned} \quad \Rightarrow \quad \dot{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} z, \quad \lambda_{1,2} = \pm i$$

But if $\varepsilon > 0$, non linear system has trajectories spiralling into the origin.

If $\varepsilon < 0$, then trajectories spiral out of origin.

Hartman Grobman Theorem

Formally state what one can conclude in neighborhood of hyperbolic equ. pts.

Let X, Y be 2 sets & $g: X \rightarrow Y$ a mapping of X into Y .

Def $g(X) = \{g(x), x \in X\}$ is image of X under g .

Def. g is a 1-1 (one to one) mapping of X into Y if for each $y \in Y$, $\overrightarrow{g}(y)$ consists of at most one element of X , i.e. $g(x_1) \neq g(x_2)$ whenever $x_1 \neq x_2$, $x_1, x_2 \in X$.

Def. $g: X \rightarrow Y$ is a mapping of X onto Y if $g(X) = Y$.

Def. g is a homeomorphism if it is a continuous, 1-1 & onto map with a continuous inverse.

diffeomorphism \rightarrow differentiable instead of continuous.

$$\dot{x} = f(x), \quad f(x^*) = 0 \quad (1)$$

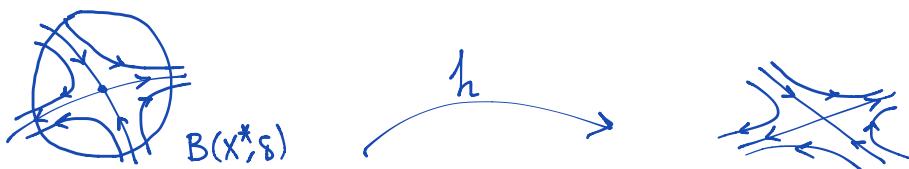
$$\dot{z} = Az, \quad A = \frac{\partial f}{\partial x}(x^*) \quad (2)$$

Theorem (Hartman-Grobman)

Suppose x^* is a hyperbolic eq. pt. of (1). Then there exists a homeomorphism

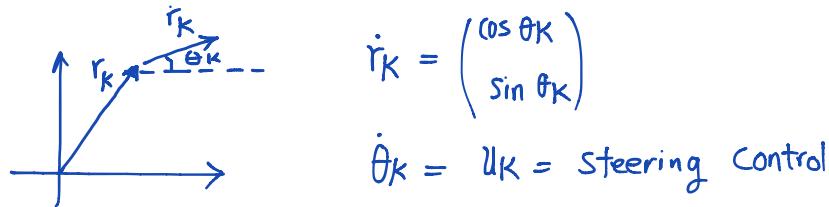
$$h: B(x^*, \delta) \rightarrow \mathbb{R}^2$$

taking trajectories of (1) and mapping them into trajectories of (2).



Planar collective motion problem

consider N agents (particles) that move in the plane at constant (unit) speed. For $k=1, \dots, N$



Consider steering law and θ_k dynamics only

$$\dot{\theta}_k = \sum_{j=1}^N a_{kj} \sin(\theta_j - \theta_k) = u_k$$

Equilibria correspond to $\sum_{j=1}^N a_{kj} \sin(\theta_j - \theta_k) = 0, \quad k=1, \dots, N$

Example : There are many eq. pts. for the system.



Example : complete graph with unit weight , $N=3$

$$\begin{aligned}\dot{\theta}_1 &= \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1) \\ &= \sin \frac{2\pi}{3} + \sin \frac{-2\pi}{3} = 0\end{aligned}$$

Example : complete graph with unit weight , $N=4$

$$\begin{aligned}\dot{\theta}_1 &= \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1) + \sin(\theta_4 - \theta_1) \\ &= \sin \frac{\pi}{2} + \sin \pi + \sin \frac{3\pi}{2} = 0\end{aligned}$$

N=3

$$\dot{\theta}_1 = \sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1)$$

$$\dot{\theta}_2 = \sin(\theta_1 - \theta_2) + \sin(\theta_3 - \theta_2)$$

$$\dot{\theta}_3 = \sin(\theta_1 - \theta_3) + \sin(\theta_2 - \theta_3)$$

$$\text{Let } \phi_1 = \theta_2 - \theta_1, \phi_2 = \theta_3 - \theta_2 \Rightarrow \phi_1 + \phi_2 = \theta_3 - \theta_1$$

$$\Rightarrow \begin{cases} \dot{\phi}_1 = -2\sin\phi_1 + \sin\phi_2 - \sin(\phi_1 + \phi_2) \\ \dot{\phi}_2 = \sin\phi_1 - 2\sin\phi_2 - \sin(\phi_1 + \phi_2) \end{cases}$$

equilibria: $(0,0), (0,\pi), (\pi,0), (\pi,\pi), \left(\frac{2\pi}{3}, \frac{2\pi}{3}\right), \left(\frac{2\pi}{3}, -\frac{2\pi}{3}\right)$ (symmetry)

$$\frac{\partial f}{\partial \phi} = \begin{pmatrix} -2\cos\phi_1 - \cos(\phi_1 + \phi_2) & \cos\phi_2 - \cos(\phi_1 + \phi_2) \\ \cos\phi_1 - \cos(\phi_1 + \phi_2) & -2\cos\phi_2 - \cos(\phi_1 + \phi_2) \end{pmatrix}$$

$$\left. \frac{\partial f}{\partial \phi} \right|_{(0,0)} = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix} \quad \text{stable node}$$

$$\left. \frac{\partial f}{\partial \phi} \right|_{(0,\pi)} = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \quad \text{saddle}$$

$$\left. \frac{\partial f}{\partial \phi} \right|_{\left(\frac{2\pi}{3}, \frac{2\pi}{3}\right)} = \begin{pmatrix} -2\left(\frac{-1}{2}\right) - \left(\frac{-1}{2}\right) & \frac{1}{2} - \left(-\frac{1}{2}\right) \\ 0 & -2\left(\frac{-1}{2}\right) - \left(\frac{-1}{2}\right) \end{pmatrix} \quad \begin{aligned} \cos \frac{2\pi}{3} &= -\frac{1}{2} \\ \cos \frac{4\pi}{3} &= -\frac{1}{2} \end{aligned}$$
$$= \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \quad \text{unstable node}$$

Phase portrait with matlab

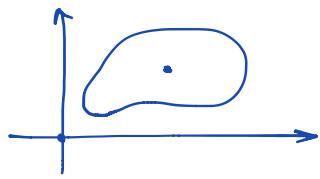
Paper to read:

- Stabilization of planar collective motion: all to all communication 2007
- Stabilization of planar collective motion with limited communication 2008

Predator-Prey system analysis

$$\dot{P} = c\alpha PN - qP$$

$$\dot{N} = rN - \alpha PN$$



recall: equilibria $(0,0)$, $(P^*, N^*) = (\frac{r}{\alpha}, \frac{q}{c\alpha})$

linearize about (P^*, N^*) :

$$x = \begin{pmatrix} P \\ N \end{pmatrix}, \quad \left. \frac{\partial f}{\partial x} \right|_{x=x^*} = \begin{pmatrix} c\alpha N - q & c\alpha P \\ -\alpha N & r - \alpha P \end{pmatrix} \Big|_{(P^*, N^*)} = (P^*, N^*)$$

$$= \begin{pmatrix} 0 & c\alpha \frac{r}{\alpha} \\ -\alpha \frac{q}{c\alpha} & 0 \end{pmatrix} = \begin{pmatrix} 0 & cr \\ -q/c & 0 \end{pmatrix}$$

$$0 = \det \begin{pmatrix} \lambda & cr \\ -q/c & \lambda \end{pmatrix} \Rightarrow \lambda^2 + rq = 0 \Rightarrow \lambda = \pm \sqrt{qr} i$$

cannot draw conclusions about the behavior of the nonlinear system near this equilibrium from this analysis.

Theorem asserts it is possible to continuously deform trajectories of nonlinear system into linear system (a single map works for continuum of trajectories in neighborhood of x^*)!

Def. The state of $(*)$ at time t , starting from x at time $t=0$ is denoted by $\phi_t(x)$ and is called the flow.

So if $x \in B(x^*, \delta)$ and $\phi_t(x) \in B(x^*, \delta)$ then

$$h(\phi_t(x)) = e^{At} h(x) \quad \text{or} \quad \phi_t(x) = h^{-1}(e^{At} h(x))$$

$$\begin{array}{ccc} x & \xrightarrow{h} & h(\phi_t(x)) = e^{At} h(x) \\ \phi_t(x) & & \end{array}$$

Qualitative behavior of nonlinear systems in vicinity of an isolated eq. pt. is determined by linearization if linearization has no imaginary axis eig's.

Limit cycles (sect 2.4)

Def. Periodic solution: $x(t+T) = x(t) \quad \forall t \geq 0$, some $T > 0$

Def. A closed orbit (periodic orbit) γ of a dynamical system is the image on the phase plane of a (nontrivial) periodic solution.

$$\gamma = \{ z \in \mathbb{R}^2 \mid z = x(t) = x(t+T), 0 \leq t < T \}$$


Thus $\gamma \subset \mathbb{R}^2$ is a closed orbit if γ is not an eq. pt. & $\exists T < \infty$ s.t. for each $x \in \gamma$, $\phi_{nT}(x) = x$, $\forall n \in \mathbb{Z}$.

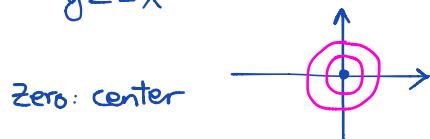
Def. A limit cycle is an isolated periodic orbit.

Note. Harmonic oscillator is not a limit cycle (not isolated)

↪ not robust since since eig's are on imaginary axis

(amplitude depends on initial conditions)

Harmonic oscillator equations: $\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x \end{aligned}$, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $\lambda_{\pm} = \pm i$



- Amplitude of a limit cycle is independent of initial conditions.*

* very intriguing & important.

Non linear oscillators are robust to perturbations.

- Non linear oscillations used in engineering - all over the place to create regular cycles.
- Nonlinear oscillations obscured in nature, most especially in biology where regular (robust) rhythms are critical (eg. in the heart).

- one important model is Van der Pol oscillator.

This was described by Balthasar Van der Pol in the 1920's when he was working at Philips Co. in the Netherlands as an electrical engineer. He designed a circuit that produces a limit cycle!

- The Van der Pol oscillator is an example of a nonlinear oscillator: System dissipates energy until reaches some threshold sufficiently close to its equilibrium where energy is added to the system. (e.g. balance between energy addition & dissipation).

- If the limit cycle is not smooth referred to as "relaxation oscillators".

- Van der Pol built an electrical circuit with a "triode vacuum tube" which has resistive properties that change with current.
(low current: - resistance, high current: + resistance)

- Many electronic relaxation oscillators store energy in a capacitor until it reaches a threshold voltage sufficiently close to the power supply. Then the capacitor can be quickly discharged. These types of circuits were used as time base in early oscilloscopes & television receivers.
- Van der Pol made the connection between the rhythm of the heart and coupled nonlinear oscillators.
- Great deal of work using such models to explain the spontaneous & repetitions firing of neurons.
- Models of circadian rhythm (biological clocks)
(the above biological examples usually consider many such oscillators with coupling which can be simplified to the coupled oscillator equations of last week)
- Single d.o.f oscillators (like Van der Pol) occur in models of
 - wind induced oscillations of buildings due to "vortex shedding"
 - aerelastic flutter problems
 - stability of tracked rubber tire vehicles
 - some chemical reactions

Van der Pol oscillator (single d.o.f oscillator with nonlinear damping)

$$\ddot{y} - \epsilon(1-y^2)\dot{y} + y = 0 \quad \epsilon > 0$$

To put this in form $\dot{x}=f(x)$, let $x_1=y$ and $x_2=\dot{y}$, then

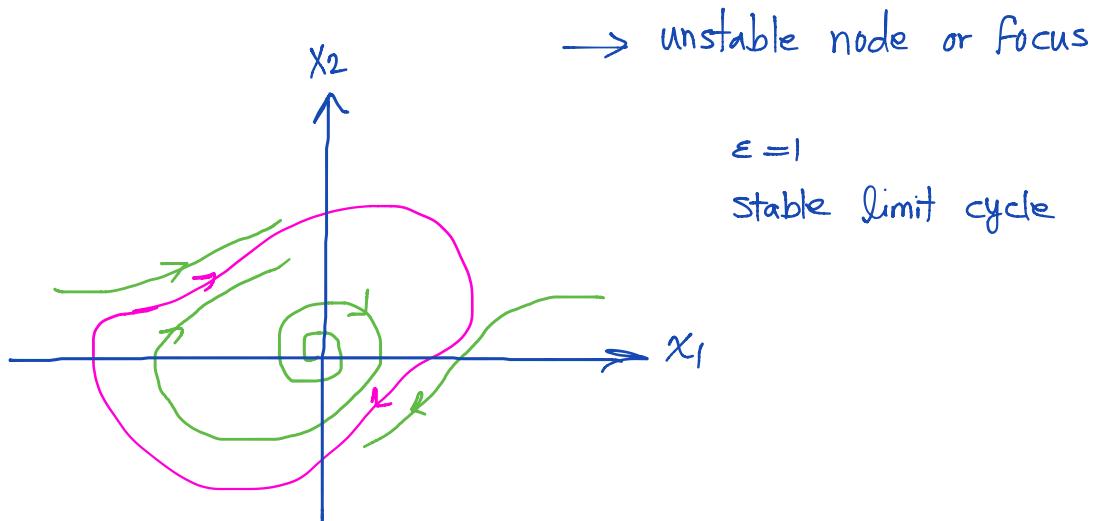
$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -x_1 + \epsilon(1-x_1^2)x_2$$

can see that for $|x_1| < 1$ negative damping
 $|x_1| > 1$ Positive damping

Equilibrium only at $x_1 = x_2 = 0$. Linearization at $(0,0)$:

$$\frac{\partial f}{\partial x} \Big|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ -1 & \varepsilon \end{pmatrix} \quad \rightarrow \quad \lambda(\lambda - \varepsilon) + 1 = 0 \\ \lambda \pm = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4}}{2}$$



consider Van der Pol oscillator in reverse time : $\tau = -t$

$$\frac{dx_1}{d\tau} = \frac{dx_1}{dt} \cdot \frac{dt}{d\tau} = \dot{x}_1(-1) = -x_2$$

$$\frac{dx_2}{d\tau} = \frac{dx_2}{dt} \cdot \frac{dt}{d\tau} = \dot{x}_2(-1) = x_1 - \varepsilon(1 - x_1^2)x_2$$

This gives same pictures with arrows reversed (unstable limit cycle)

Bendixon Criterion

use to rule out existence of a limit cycle

Let D be a simply connected region in \mathbb{R}^2 .

(A simply connected region has no holes. i.e. it is contractable to a point. For every simple closed curve " C " in D , the interior of C is also a subset of D)

If $\operatorname{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign in D , then D contains no closed orbit of $\dot{x} = f(x)$.

proof. By Green's Theorem

on any orbit $\frac{dx_2/dt}{dx_1/dt} = \frac{dx_2}{dx_1} = \frac{f_2(x_1, x_2)}{f_1(x_1, x_2)}$ or $f_2 dx_1 - f_1 dx_2 = 0$.

So on a closed orbit γ we have:

$$\int_{\gamma} f_2(x_1, x_2) dx_1 - f_1(x_1, x_2) dx_2 = 0$$

By Green's thm, $\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right)(x_1, x_2) dx_1 dx_2 = 0$
 $S = \text{interior of } \gamma$

which is not possible since $\operatorname{div} f$ is always + or -.

Example. Van der Pol oscillator

$$f_1 = x_2 \quad \rightarrow \operatorname{div} f = \varepsilon(1-x_1^2)$$
$$f_2 = -x_1 + \varepsilon(1-x_1^2)x_2$$

so no L.C. in region st $|x_1| < 1$, since then $\operatorname{div} f > 0$.

Example. $\dot{x}_1 = x_2 + x_1 x_2^2$
 $\dot{x}_2 = -x_1 + x_1^2 x_2$

equilibrium: $x_2(1+x_1 x_2) = 0 \Rightarrow x_1 = x_2 = 0$
 $x_1(-1+x_1 x_2) = 0$

Linearization at $(0,0)$:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \rightarrow \lambda_{1,2} = \pm i$$

$$\operatorname{div} F = x_1^2 + x_2^2 > 0 \text{ on every region } D.$$

\Rightarrow No L.C. anywhere!

Example. Predator-Prey

$$\dot{P} = c a P N - q P = f_1(P, N)$$

$$\dot{N} = r N - a P N = f_2(P, N)$$

$$\operatorname{div} F = c a N - q + a P = a \left[c \left(N - \frac{q}{c a} \right) - \left(P - \frac{r}{a} \right) \right]$$

$$> 0 \quad \text{if } c \left(N - \frac{q}{c a} \right) > P - \frac{r}{a}$$

$$< 0 \quad \text{if } <$$

\Rightarrow Cannot rule out periodic orbit.

Example. $\dot{x}_1 = x_2 + \alpha x_1 (\beta^2 - x_1^2 - x_2^2) \quad \alpha > 0$
 $\dot{x}_2 = -x_1 + \alpha x_2 (\beta^2 - x_1^2 - x_2^2)$

equilibrium $(0,0)$

$$\text{Linearization} \quad \frac{\partial f}{\partial x} \Big|_{(0,0)} = \begin{pmatrix} \alpha\beta^2 & 1 \\ -1 & \alpha\beta^2 \end{pmatrix}$$

eigenvalues $\lambda_{1,2} = \alpha\beta^2 \pm i$ unstable focus for $\alpha > 0$

$$\text{Define } r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \tan^{-1}\left(\frac{x_2}{x_1}\right)$$

$$\dot{r} = \alpha r (\beta^2 - r^2), \quad \dot{\theta} = -1 \quad (*)$$

$$x_1 = r \cos \theta, \quad x_2 = r \sin \theta$$

$$\dot{r} = \frac{1}{2r} (2\dot{x}_1 x_1 + 2\dot{x}_2 x_2) = \frac{1}{r} (\alpha r^2 (\beta^2 - r^2)) = \alpha r (\beta^2 - r^2)$$

$$\text{use } \frac{d}{dx} \tan^{-1} u = \frac{1}{1+u^2} \frac{du}{dx}$$

$$\begin{aligned} \dot{\theta} &= \frac{1}{1 + (\dot{x}_2/x_1)^2} \frac{d(x_2/x_1)}{dt} = \frac{1}{1 + (\dot{x}_2/x_1)^2} \left(\frac{\dot{x}_2}{x_1} - \frac{x_2}{x_1^2} \dot{x}_1 \right) \\ &= \frac{x_1 \dot{x}_2 - x_2 \dot{x}_1}{x_1^2 + x_2^2} = \frac{-x_1^2 - x_2^2}{x_1^2 + x_2^2} = -1 \end{aligned}$$

when $r = \beta \rightarrow \dot{r} = 0, \dot{\theta} = -1 \rightarrow$ closed orbit

can integrate (*) to get:

$$r(t) = \beta \left(1 + \left(\frac{\beta}{r(0)} - 1 \right) e^{-2\beta\alpha t} \right)^{-1/2}, \quad r(0) = 0$$

$$\theta(t) = \theta(0) - t$$

All non zero initial conditions converge to limit cycle.

$$\underline{\alpha = \beta = 1}$$

